

TAC-1999-018
SISSA 124/99/EP

Neutrino oscillations in the early universe: How large lepton asymmetry can be generated?

A.D. Dolgov^{1 2}, S.H. Hansen³

Teoretisk Astrofysik Center

Juliane Maries Vej 30, DK-2100, Copenhagen, Denmark

S. Pastor⁴

SISSA-ISAS and INFN, Sezione di Trieste

Via Beirut 2-4, I-34013 Trieste, Italy

D.V. Semikoz⁵

Max-Planck-Institut für Physik (Werner-Heisenberg-Institut)

Föhringer Ring 6, 80805 München, Germany

and

Institute of Nuclear Research of the Russian Academy of Sciences

60th October Anniversary Prospect 7a, Moscow 117312, Russia

Abstract

The lepton asymmetry that could be generated in the early universe through oscillations of active to sterile neutrinos is calculated (almost) analytically for small mixing angles, $\sin 2\theta < 10^{-2}$. It is shown that for a mass squared difference, $\delta m^2 = -1 \text{ eV}^2$ it may rise at most by 6 orders of magnitude from the initial “normal” value $\sim 10^{-10}$, since the back-reaction from the refraction index terminates this rise while the asymmetry is still small. Only for very large mass differences, $|\delta m^2| \sim 10^9 \text{ eV}^2$, the lepton asymmetry could reach a significant magnitude exceeding 0.1.

¹Also: ITEP, Bol. Cheremushkinskaya 25, Moscow 113259, Russia.

²e-mail: dolgov@tac.dk

³e-mail: sthansen@tac.dk

⁴e-mail: pastor@sissa.it

⁵e-mail: semikoz@ms2.inr.ac.ru

Neutrino oscillations in the early universe differ from e.g. solar oscillations in two important aspects. First, one cannot neglect neutrino annihilation or scattering in the medium for a rather large and physically interesting range of parameters (mixing angle, $\sin 2\theta$, and mass squared difference, δm^2). These processes break the coherence of neutrino propagation and that is why considerations in terms of wave functions become impossible and one has to use the density matrix formalism [1, 2]. Kinetic equations for the density matrix of oscillating neutrinos with the account of the so called second order effects (proportional to the second power of the Fermi coupling constant, G_F) were derived in refs. [1, 3, 4]. Second, neutrino oscillations in the primeval plasma could modify the plasma properties, in particular the refraction index, and this in turn would influence the oscillations, so that the problem becomes highly nonlinear. The refraction index of oscillating neutrinos in the cosmic plasma was calculated in ref. [5]. An important feature of the refraction index is that it contains terms proportional to the charge asymmetry of the cosmic plasma. The effective potential of a standard neutrino of flavor a can be written as

$$V_{eff}^a = \pm C_1 \eta G_F T^3 + C_2^a \frac{G_F^2 T^4 E}{\alpha}, \quad (1)$$

where E is the neutrino energy, T is the temperature of the plasma, $G_F = 1.166 \cdot 10^{-5} \text{ GeV}^{-2}$ is the Fermi coupling constant, $\alpha = 1/137$ is the fine structure constant, and the signs “ \pm ” refer to anti-neutrinos and neutrinos respectively (this choice of sign describes the helicity state, negative for ν and positive for $\bar{\nu}$). According to ref. [5] the coefficients C_j are: $C_1 \approx 0.95$, $C_2^e \approx 0.61$ and $C_2^{\mu,\tau} \approx 0.17$. These values are true in the case of thermal equilibrium, and otherwise these coefficients are some integrals over the distribution functions. In ref. [5] the coefficient C_1 was calculated using the present value of the asymmetry η , which differs from its value in the early Universe (before e^+e^- annihilation increased the photon temperature) by a factor 11/4. In our calculations we took $C_1 = 0.345$. The charge asymmetry, η , is defined as the ratio of the difference between particle-antiparticle number densities to the number density of photons. The individual contributions to η from different particle species are the

following:

$$\eta = 2\eta_{\nu_e} + \eta_{\nu_\mu} + \eta_{\nu_\tau} + \eta_e - \eta_n/2 \text{ (for } \nu_e), \quad (2)$$

$$\eta = 2\eta_{\nu_\mu} + \eta_{\nu_e} + \eta_{\nu_\tau} - \eta_n/2 \text{ (for } \nu_\mu), \quad (3)$$

and η for ν_τ is obtained from eq. (3) by the interchange $\mu \leftrightarrow \tau$.

The different magnitude of the refraction indices for neutrinos and anti-neutrinos may result in more favorable conditions for $\nu_a \rightarrow \nu_s$ oscillations compared with $\bar{\nu}_a \rightarrow \bar{\nu}_s$ oscillation, where ν_a is an active neutrino, $a = e, \mu, \tau$, and ν_s is a sterile one. Since more ν_a than $\bar{\nu}_a$ would be transformed into sterile ones, the lepton asymmetry in the sector of active neutrinos would rise and this would further amplify the process. The possibility of such instability was noticed in ref. [6], but there it was found, on the basis of simplified considerations, that the rise is stabilized when non-linear terms in the refraction index become non-negligible, and thus it was concluded, that “no large chemical potential will be generated in any point of the parameter space”. A similar statement of a small asymmetry was made in ref. [7]. This conclusion was reconsidered in ref. [8] (see also refs. [9, 10]) where it was argued that a very large asymmetry, even close to 1, may be generated by the oscillations. An even more striking statement was made in ref. [11], that the asymmetry may not only be large, but may have a chaotic sign, so that even domains with different signs of the asymmetry may be formed [12]. Similar results were obtained in the recent paper [13], namely, that the asymmetry may reach large values and in some ranges of parameters its sign may be chaotic, while in ref. [14] chaoticity was not observed. However, these results were derived in a simplified way after averaging some essential quantities over neutrino momenta or through a solution of momentum dependent but simplified equations. At the same time, in “brute force” numerical calculations applied to this problem, it is very difficult to distinguish between the real effect and a computational instability. Technically there is an essential difference between exact momentum dependent calculations and the momentum averaged ones. In the last case one has to solve a set of ordinary differential equations, while in the former case the

corresponding equations are integro-differential ones (see e.g. ref. [15], where exact kinetic equations were accurately solved for non-oscillating neutrinos). Momentum dependent exact numerical calculations were done in a series of papers [16]. It was shown there that the asymmetry may rise by several (3–4) orders of magnitude but still remains small in the range of parameters $|\delta m^2| < 10^{-7} \text{ eV}^2$, assuming a small initial asymmetry, 10^{-10} . Outside this range of parameters the calculations became unstable and no definite result was obtained. We extended the domain of stable computation to a somewhat larger range of parameters [17]:

$$|\delta m^2| < 10^{-6} \text{ eV}^2, \quad (4)$$

and have also not found any large generation of charge asymmetry. In the coinciding range of parameters our results are in a reasonable agreement with those of ref. [16]. The attempts to extend the range (4) maintaining the stability of the computational process demanded a huge increase of computer time, because otherwise the results were chaotic in sign and showed a quickly rising asymmetry. Thus it seems that further attempts to extend stable numerical calculations to a wider parameter range, in which the huge amplification of the asymmetry was found, will be fruitless and one should try to transform the kinetic equations for the oscillating neutrinos analytically to such a form that will permit a numerical solution. In what follows we have achieved this goal and reduced the problem to the solution of an (almost) ordinary differential equation for the neutrino charge asymmetry, which is easy to solve numerically.

Before presenting the actual calculations we will briefly describe our procedure so that it will be easier to follow it. We start from the usual equations for the evolution of the neutrino (as well as anti-neutrino) density matrix in a cosmological environment. We twice introduce new variables, first x and y given by eq. (11) so that the evolution operator in the l.h.s. of the kinetic equations would depend only on one variable x , and second, we rewrite all the equations in terms of τ given by eq. (22). It is a natural variable for the description of the behavior of the density matrix near the MSW resonance, where it takes the value $\tau = 1$, in the limit of

a vanishing contribution of charge asymmetry to the refraction index. As we see in what follows almost all coefficient functions in the kinetic equations depend only on τ except for the charge asymmetric potential V in eq. (24) that depends on two variables τ and y . Using the variable τ permits to factor out the large parameter Q in eq. (29) related to a large frequency of oscillations. This is true for neutrino mass differences above 10^{-7} eV². Quick oscillations make numerical computations very difficult. Fortunately one may analytically separate the quickly varying functions and make an expansion in terms of $1/Q$. An essential technical step is to consider charge symmetric and antisymmetric elements of the density matrix, $\rho \pm \bar{\rho}$. Antisymmetric combination directly enters the refraction index (see eqs. (1,20)) and working with symmetric and antisymmetric functions permits to derive a closed equation for the evolution of the asymmetry. As a first step we formally solve equations (14)–(17) for the antisymmetric elements of the density matrix in terms of unknown symmetric functions and the integrated charge asymmetry Z in eq. (20). In the limit of large Q the corresponding differential equations allow a simple algebraic solution. As a second step we substitute the obtained expressions into the charge symmetric equations. For the latter we find eigenfunctions that are formal solutions of these equations in the case of constant coefficients. However since the latter are not constant we obtain a system of differential equations for the coefficient functions in the expansion of the solution in terms of the eigenvectors. The equations for these coefficient functions are quite simple and can be solved numerically and analytically (both approaches give very close results). After that we substitute the found solutions, which contain an unknown charge asymmetry Z , back into the antisymmetric equations and after integration over momentum of both sides of the equation for $(\rho'_{aa} - \bar{\rho}'_{aa})$ we obtain a closed differential equation for the charge asymmetry Z . The latter is a function of a single variable $q = y\tau$ and it can be relatively easily solved numerically. Now we will describe the same in more detail.

The basic equations governing evolution of density matrix are:

$$i(\partial_t - Hp\partial_p)\rho_{aa} = F_0(\rho_{sa} - \rho_{as})/2 - i\Gamma_0(\rho_{aa} - f_{eq}) , \quad (5)$$

$$i(\partial_t - Hp\partial_p)\rho_{ss} = -F_0(\rho_{sa} - \rho_{as})/2 , \quad (6)$$

$$i(\partial_t - Hp\partial_p)\rho_{as} = W_0\rho_{as} + F_0(\rho_{ss} - \rho_{aa})/2 - i\Gamma_1\rho_{as} , \quad (7)$$

$$i(\partial_t - Hp\partial_p)\rho_{sa} = -W_0\rho_{sa} - F_0(\rho_{ss} - \rho_{aa})/2 - i\Gamma_1\rho_{sa} , \quad (8)$$

where a and s mean “active” and “sterile” respectively, $F_0 = \delta m^2 \sin 2\theta/2E$, $W_0 = \delta m^2 \cos 2\theta/2E + V_{eff}^a$, $H = \sqrt{8\pi\rho_{tot}/3M_p^2}$ is the Hubble parameter, p is the neutrino momentum, and f_{eq} is the equilibrium Fermi distribution function:

$$f_{eq} = [\exp(E/T) + 1]^{-1} . \quad (9)$$

More precisely instead of the equilibrium function f_{eq} one should use the one with a non-zero chemical potential, because scattering and annihilation processes do not change lepton number. However if the asymmetry is not large the difference between them is not important for our calculations.

The anti-neutrino density matrix satisfies the similar set of equations with the opposite sign of the antisymmetric term in V_{eff}^a and with a slight difference in damping factors that is proportional to the lepton asymmetry.

Equations (5-8) account exactly for the first order terms described by the refraction index, while the second order terms describing coherence breaking are approximately modeled by the damping coefficients Γ_j . The latter are equal to [18]:

$$\Gamma_0 = 2\Gamma_1 = g_a \frac{180\zeta(3)}{7\pi^4} G_F^2 T^4 p . \quad (10)$$

In general the coefficient $g_a(p)$ is a momentum-dependent function, but in the approximation of neglecting $[1 - f]$ factors in the electro-weak collision rates it becomes a constant [19] that corresponds to $g_{\nu_e} \simeq 4$ and $g_{\nu_\mu, \nu_\tau} \simeq 2.9$ [20]. In the following we will use a more accurate value of g_a , which comes from the thermal average of the complete electro-weak rates (with factors $[1 - f]$ included), which we calculated numerically

from our Standard Model code [15]. This gives us $g_{\nu_e} \simeq 3.56$ and $g_{\nu_\mu, \mu_\tau} \simeq 2.5$. The indices sub-0 are here prescribed to the coefficient functions to distinguish them from the similar ones after dividing by Hx (see below).

It is convenient to introduce new variables:

$$x = m_0 R(t) \text{ and } y = pR(t) , \quad (11)$$

where $R(t)$ is the cosmological scale factor so that $H = \dot{R}/R$ and m_0 is an arbitrary mass (just normalization), we choose $m_0 = 1$ MeV. In the approximation that we will work, we assume that $\dot{T} = -HT$, so that we can take $R = 1/T$. In terms of these variables the differential operator $(\partial_t - Hp\partial_p)$ transforms to $Hx\partial_x$. We will normalize the density matrix elements⁶ to the equilibrium distribution:

$$\rho_{aa} = f_{eq}(y)[1 + a(x, y)], \quad \rho_{ss} = f_{eq}(y)[1 + s(x, y)] , \quad (12)$$

$$\rho_{as} = \rho_{sa}^* = f_{eq}(y)[h(x, y) + il(x, y)] , \quad (13)$$

and the neutrino mass difference δm^2 to eV².

As the next step we will take the sum and difference of eqs. (5)-(8) for ν and $\bar{\nu}$. The corresponding equations have the following form:

$$s'_\pm = Fl_\pm , \quad (14)$$

$$a'_\pm = -Fl_\pm - 2\gamma_+ a_\pm - 2\gamma_- a_\mp , \quad (15)$$

$$h'_\pm = Ul_\pm - VZl_\mp - \gamma_+ h_\pm - \gamma_- h_\mp , \quad (16)$$

$$l'_\pm = \frac{F}{2}(a_\pm - s_\pm) - Uh_\pm + VZh_\mp - \gamma_+ l_\pm - \gamma_- l_\mp , \quad (17)$$

where $a_\pm = (a \pm \bar{a})/2$ etc, and the prime means differentiation with respect to x . We have used $W = U \pm VZ$, $\gamma = \Gamma_1/Hx$, and $\gamma_\pm = (\gamma \pm \bar{\gamma})/2$, where γ_- parameterizes

⁶Other authors find it convenient to express this density matrix formalism in terms of Pauli matrices and a polarization vector, $\vec{P} = (P_x, P_y, P_z)$, such that:

$$\rho \equiv \frac{P_0}{2} \left[1 + \vec{P} \cdot \vec{\sigma} \right] ,$$

in such a way that $P_0 P_z = 1$ means that all the neutrinos are ν_e , and we have $P_0 P_x = f_{eq} h$, $P_0 P_y = -f_{eq} l$, $P_0 P_z = f_{eq}(a - s)$ and $P_0 = f_{eq}(2 + a + s)$.

the difference of interaction rates between neutrino and anti-neutrinos, which is proportional to the neutrino asymmetry. With the approximation⁷ $\rho_{tot} \simeq 10.75\pi^2 T^4/30$, the expressions for U , V , and Z become:

$$U = 1.12 \cdot 10^9 \cos 2\theta \delta m^2 \frac{x^2}{y} + 26.2 \frac{y}{x^4}, \quad (18)$$

$$V = \frac{29.6}{x^2}, \quad (19)$$

$$Z = 10^{10} \left(\eta_o - \int \frac{dy}{4\pi^2} y^2 f_{eq} a_- \right), \quad (20)$$

where η_o is the asymmetry of the other particle species (see eqs. (2,3)) normalized in the same way as the neutrino asymmetry (the second term in (20)). Here we have implicitly assumed that $\nu_a = \nu_e$.

We can use total leptonic charge conservation to determine γ_- , but as we see in what follows, the γ_- -terms are either sub-dominant or not important, so we do not need a concrete expression for γ_- . The most unpleasant contribution, which makes it so difficult to solve the symmetric equations numerically, comes from the term containing an *integral* over momentum of the difference $(\rho_{aa} - \bar{\rho}_{aa})$ with a large coefficient. If the lepton asymmetry is sufficiently small, $\eta < 10^{-7}$, this term can be neglected in the symmetric equations, but when it is large its back-reaction on the rise of the asymmetry is quite important. All other (asymmetric)² terms in the symmetric equations, e.g. $\gamma_- a_-$, can always be neglected. The only essential terms are proportional to VZ because they enter with a numerically large coefficient (see below).

Let us introduce some more notations. Since the asymmetric term a_- enters the expression (20) with the coefficient 10^{10} we introduce capital letters for the renormalized asymmetric functions:

$$S = 10^{10} s_-, \quad A = 10^{10} a_-, \quad H = 10^{10} h_-, \quad \text{and} \quad L = 10^{10} l_- . \quad (21)$$

⁷This approximation is valid for high temperatures $T > 1$ MeV. In our case for $\delta m^2 = -1$ eV² and any small $\sin 2\theta \ll 1$ we deal with temperatures above 10 MeV for all essential momenta. And only for $|\delta m^2| < 10^{-6}$ eV² one will need to take into account a more accurate expression for ρ_{tot} .

We will also introduce a new variable:

$$\tau = \xi x^3/y , \quad (22)$$

where $\xi \approx 6.5 \cdot 10^3 \sqrt{|\delta m^2| \cos 2\theta}$ so that U vanishes at $\tau = 1$ or, in other words, the MSW resonance takes place at $\tau = 1$ if $\delta m^2 < 0$ and if the contribution from the asymmetric part, VZ , can be neglected. We will divide everything by the factor $M = 1.12 \cdot 10^9 \cos 2\theta |\delta m^2| x^2/y$, so that the coefficient functions now become:

$$F = -\tan 2\theta \approx -\sin 2\theta, \quad U = 1/\tau^2 - 1, \quad \gamma = \delta/\tau^2 , \quad (23)$$

where $\delta \approx 1/135$ is a small coefficient. In what follows we will often use the notation $\gamma \equiv \gamma_+$. The asymmetric potential, V/M , in terms of these variables has the form:

$$V = 3.3 \cdot 10^{-3} (\cos 2\theta \delta m^2)^{-1/3} y q^{-4/3} . \quad (24)$$

where we have introduced $q \equiv y\tau$. The equations for the asymmetric functions can now be written as:

$$S'/Q = FL , \quad (25)$$

$$A'/Q = -FL - 2\gamma A - 2 \cdot 10^{10} \gamma_- a_+ , \quad (26)$$

$$H'/Q = UL - \gamma H - 10^{10} VZ l_+ - 10^{10} \gamma_- h_+ , \quad (27)$$

$$L'/Q = \frac{F}{2} (A - S) - UH - \gamma L + 10^{10} VZh_+ - 10^{10} \gamma_- l_+ , \quad (28)$$

where prime now means derivative with respect to τ and:

$$Q \approx 5.6 \cdot 10^4 \sqrt{|\delta m^2 \cos 2\theta|} . \quad (29)$$

Due to conservation of leptonic charge the integrated contribution of the last two terms in the r.h.s. of eq. (26) vanishes:

$$\int dy y^2 f_{eq}(y) (\gamma A + 10^{10} \gamma_- a_+) = 0 . \quad (30)$$

In the equations for H' and L' we neglect the terms proportional to $10^{10} \gamma_- \sim Z$ as well as $F(A - S)/2$ because they are small in comparison with $10^{10} VZ \sim 10^7 Z$.

We will solve these equations in the limit of large $Q \gg 1$, the corrections generally being of the order of $1/Q$. A formal solution of equations (27) and (28) (in terms of unknown functions Z , h_+ , and l_+) is:

$$L = \frac{10^{10} V Z}{\gamma^2 + U^2} (l_+ U + h_+ \gamma) , \quad (31)$$

$$H = \frac{10^{10} V Z}{\gamma^2 + U^2} (-l_+ \gamma + h_+ U) . \quad (32)$$

This approximation works if $Z(q)$ does not decrease too fast with increasing q and is even better justified if $Z(q)$ is a rising function of q .

These solutions can now be inserted into the set of equations for the symmetric function, which can be written in the matrix form as $\mathcal{V}' = \mathcal{M}\mathcal{V}$:

$$\begin{pmatrix} s'_+ \\ a'_+ \\ h'_+ \\ l'_+ \end{pmatrix} = Q \begin{pmatrix} 0 & 0 & 0 & F \\ 0 & -2\gamma & 0 & -F \\ 0 & 0 & -\tilde{\gamma} & \tilde{U} \\ -F/2 & F/2 & -\tilde{U} & -\tilde{\gamma} \end{pmatrix} \begin{pmatrix} s_+ \\ a_+ \\ h_+ \\ l_+ \end{pmatrix} . \quad (33)$$

Here we have used:

$$\tilde{U} = U (1 - D^2/\sigma^2) \text{ and } \tilde{\gamma} = \gamma (1 + D^2/\sigma^2) , \quad (34)$$

with $D^2 = Z^2 V^2$ and $\sigma^2 = \gamma^2 + U^2$.

We will solve this system of equations expanding the solution in terms of the eigenvectors of the corresponding matrix \mathcal{M} in the r.h.s. of eq. (33) with the coefficients b_j that are not constants (as they would be if \mathcal{M} was a constant matrix), but functions of τ . For the functions b_j we will obtain a set of differential equation that can easily be solved numerically and even analytically in the limit of a small $\sin 2\theta$. The matrix \mathcal{M} has 4 eigenvectors with the eigenvalues:

$$\mu_1 \approx -\frac{F^2 \tilde{\gamma}}{2\tilde{\sigma}^2} \quad (35)$$

$$\mu_2 \approx -2\gamma + \frac{F^2(2\gamma - \tilde{\gamma})}{2[(2\gamma - \tilde{\gamma})^2 + \tilde{U}^2]} \quad (36)$$

$$\mu_{3,4} \approx -\tilde{\gamma} \pm i\tilde{U} \quad (37)$$

where $\tilde{\sigma}^2 = \tilde{\gamma}^2 + \tilde{U}^2$. The correction to μ_2 is of the order of F^2 when $D^2 - \sigma^2 \neq 0$. When the latter quantity is close to zero, the correction may be of the order of F .

The matrix elements of the symmetric density matrix can be expressed through the four new functions $b_j(\tau)$ as:

$$s_+ = -b_0 + b_1 \frac{F^2}{4\sigma^2} - F \frac{\sigma^2 - D^2}{\sigma^2 \tilde{\sigma}^2} \left[\tilde{\gamma} (b_2 \cos \Omega - b_3 \sin \Omega) - \tilde{U} (b_2 \sin \Omega + b_3 \cos \Omega) \right] , \quad (38)$$

$$a_+ = -b_0 \frac{F^2(\sigma^2 + D^2)}{4\sigma^2 \tilde{\sigma}^2} + b_1 \frac{\sigma^2 - D^2}{\sigma^2} - \frac{F}{\sigma^2} [\gamma (b_2 \cos \Omega - b_3 \sin \Omega) + U (b_2 \sin \Omega + b_3 \cos \Omega)] , \quad (39)$$

$$h_+ = b_0 \frac{FU(\sigma^2 - D^2)}{2\sigma^2 \tilde{\sigma}^2} + b_1 \frac{FU}{2\sigma^2} + \frac{\sigma^2 - D^2}{\sigma^2} (b_2 \sin \Omega + b_3 \cos \Omega) , \quad (40)$$

$$l_+ = b_0 \frac{F\gamma(\sigma^2 + D^2)}{2\sigma^2 \tilde{\sigma}^2} - b_1 \frac{F\gamma}{2\sigma^2} + \frac{\sigma^2 - D^2}{\sigma^2} (b_2 \cos \Omega - b_3 \sin \Omega) , \quad (41)$$

where $\Omega' = Q\tilde{U}$. To the leading order in the small parameter F the function b_0 satisfies the equation⁸

$$b_0' = -\frac{QF^2\tilde{\gamma}}{2\tilde{\sigma}^2} b_0 . \quad (42)$$

We usually neglect terms of the order of F^2 , but the one above contains the large factor Q and is therefore taken into account.

The function b_1 is small, $b_1 \sim F^2$, and can be neglected. The functions $(b_{2,3}\psi)$ satisfy the equations:

$$(b_2\psi)' = -Q\tilde{\gamma} (b_2\psi) - \frac{b_0}{2} \left[(\tilde{\alpha}\psi)' \sin \Omega + (\tilde{\beta}\phi)' \cos \Omega \right] , \quad (43)$$

$$(b_3\psi)' = -Q\tilde{\gamma} (b_3\psi) - \frac{b_0}{2} \left[(\tilde{\alpha}\psi)' \cos \Omega - (\tilde{\beta}\phi)' \sin \Omega \right] , \quad (44)$$

where $\psi = (\sigma^2 - D^2)/\sigma^2$, $\phi = (\sigma^2 + D^2)/\sigma^2$, $\tilde{\alpha} = FU/\tilde{\sigma}^2$, and $\tilde{\beta} = F\gamma/\tilde{\sigma}^2$. The initial conditions are $b_0(0) = -1$ and $b_{1,2,3}(0) = 0$.

⁸Note that this equation can be written as the evolution equation for the sterile neutrino (and anti-neutrino) density in momentum space as shown in ref. [9] (eq. (93)) or ref. [19] (eq. (72)).

When $D = 0$ it is straightforward to solve for the b -functions numerically. From fig. 1 it is clear that $b_{1,2,3}$ are very small except near the resonance. For momentum $y = 1$ and parameters $\sin 2\theta = 10^{-3}$ and $\delta m^2 = -1$, we see that the function b_0 follows the curve $\exp[-\kappa \cdot (\arctan(2(q-1)/\delta) + \pi/2)]$ to a high accuracy (remember that $q = y\tau$). Here κ goes like $\sin^2 2\theta$ for small mixing angles.

The last two equations (43,44) can be solved as:

$$b_2 \psi = -\frac{1}{2} \int_0^\tau d\tau_1 e^{-\Gamma_3(\tau)+\Gamma_3(\tau_1)} b_0(\tau_1) \left[(\tilde{\alpha}\psi)'_1 \sin \Omega_1 + (\tilde{\beta}\phi)'_1 \cos \Omega_1 \right], \quad (45)$$

$$b_3 \psi = -\frac{1}{2} \int_0^\tau d\tau_1 e^{-\Gamma_3(\tau)+\Gamma_3(\tau_1)} b_0(\tau_1) \left[(\tilde{\alpha}\psi)'_1 \cos \Omega_1 - (\tilde{\beta}\phi)'_1 \sin \Omega_1 \right], \quad (46)$$

where $\Gamma'_3 = Q\tilde{\gamma}$ and sub-1 means that the argument of the corresponding function is τ_1 . When substituting these results into eqs. (40,41) and integrating by parts we obtain:

$$h_+ = \frac{Q}{2} \int_0^\tau d\tau_1 e^{-\Delta\Gamma} b_0(\tau_1) \left[F \sin \Delta\Omega - \frac{b'_0}{Qb_0} (\tilde{\beta}_1 \phi_1 \sin \Delta\Omega - \tilde{\alpha}_1 \psi_1 \cos \Delta\Omega) \right], \quad (47)$$

$$l_+ = -\frac{Q}{2} \int_0^\tau d\tau_1 e^{-\Delta\Gamma} b_0(\tau_1) \left[F \cos \Delta\Omega - \frac{b'_0}{Qb_0} (\tilde{\alpha}_1 \psi_1 \sin \Delta\Omega + \tilde{\beta}_1 \phi_1 \cos \Delta\Omega) \right], \quad (48)$$

where $\Delta\Gamma_3 = \Gamma_3(\tau) - \Gamma_3(\tau_1)$ and $\Delta\Omega = \Omega(\tau_1) - \Omega(\tau)$. The integrals can be taken in the limit of large Q according to:

$$\int_0^\tau d\tau_1 \Phi(\tau_1) e^{-\Delta\Gamma - i\Delta\Omega} = \frac{1}{Q} \frac{\Phi(\tau)}{\tilde{\gamma} - i\tilde{U}}. \quad (49)$$

This result permits us to express the function L from eq. (31) algebraically through the lepton asymmetry Z :

$$L = -10^{10} \frac{FVZ\gamma U}{\sigma^2 \tilde{\sigma}^2} b_0 \left[1 - \frac{F^2(\sigma^2 + D^2)}{4\sigma^2 \tilde{\sigma}^2} \right]. \quad (50)$$

Substituting this result into eq. (26) and integrating it with $dy y^2 f_{eq}(y)/4\pi^2$ we finally derive the following equation governing the evolution of the lepton asymmetry $Z(q)$:

$$\frac{1}{Z} \frac{dZ}{dq} = -\delta B q^{5/3} \int_0^\infty dt \frac{t^4(t^2 - 1) f_{eq}(tq) b_0(1/t)}{\sigma^2 \tilde{\sigma}^2} \left(1 - B_1 \frac{\sigma^2 + D^2}{\sigma^2 \tilde{\sigma}^2} \right), \quad (51)$$

where:

$$B = 4.71 \cdot 10^4 (\cos 2\theta |\delta m^2|)^{1/6} \left(\frac{\sin 2\theta}{10^{-3}} \right)^2, \quad (52)$$

$$B_1 = 2.5 \cdot 10^{-7} \left(\frac{\sin 2\theta}{10^{-3}} \right)^2, \quad (53)$$

and where we have introduced the new integration variable $t = 1/\tau$, which is proportional to the neutrino momentum. If we neglect the term proportional to B_1 our evolution equation (51) coincides with the main contribution of the “static approximation” in refs. [8, 9] (see for instance eq. (65) in [19]). However the last term in eq. (51) stops the rise of the asymmetry Z earlier and at much lower values.

In the limit of a small Z we may neglect D in the r.h.s. of this equation and it can easily be integrated. If we assume that b_0 does not vary, i.e. $b_0 \equiv -1$, then the integral over dt is (approximately) proportional to $(1 - q)$, so that for $q < 1$ the asymmetry decreases, and for $q > 1$ it starts to rise. It is easy to check that with $D = 0$ the integrated rise is stronger than the decrease and thus the asymmetry rises by the factor $\exp [3.7(10^5 \sin 2\theta)^2]$ for the mass $\delta m^2 = -1$. For any $\sin 2\theta \geq 2 \cdot 10^{-5}$ there would be an enormous rise of the asymmetry.

This does not happen, however, because for a large θ the variation of b_0 should be taken into account. It changes as:

$$\Delta b_0 = b_0(\infty) - b_0(0) \approx 0.04 \left(\frac{\sin 2\theta}{10^{-3}} \right)^2 |\delta m^2|^{1/2}, \quad (54)$$

Even this relatively weak variation happens to be vitally important for the evolution of the asymmetry. The point is that the integral in the r.h.s. of eq. (51) has a resonance at $U = 0$. The contribution of this resonance into the integral is quite small for a constant b_0 , because the factor U in the numerator cancels it out since U is an odd function near the resonance. However, since b_0 experiences variation exactly at the resonance point, its variation breaks the symmetry from the positive and negative contribution of U near the resonance and the relative effect of the small variation of b_0 is enhanced by the factor $1/\delta \sim 10^2$. This effect diminishes the positive contribution

into the r.h.s. of eq. (51) and the rise of the asymmetry is strongly suppressed. In particular for $\sin 2\theta \geq 10^{-3}$ the integrated contribution of the r.h.s. becomes negative and the asymmetry decreases with respect to its initial value.

The variation of b_0 is also important because it gives an upper limit to a possible generation of lepton asymmetry. Because of leptonic charge conservation the asymmetry generated in the sector of active neutrinos must be equal to that in the sector of sterile ones. The latter is proportional to the difference of the diagonal matrix elements of the density matrix, $\Delta Z \sim \Delta(s - \bar{s})$. Since the variation of s and \bar{s} can only be positive (initial value of both is -1), $\Delta(s - \bar{s}) < \Delta(s + \bar{s})$ and the last quantity is given by the variation of b_0 . Naively taken from eq. (54) this variation is rather small for small θ . However, as we see in what follows, the variation of b_0 is rising with rising Z , so the discussed limit is not broken. On the other hand, the back-reaction from the variation of b_0 terminates the rise of the asymmetry when it is still not too large; the maximum amplification, that happens to be near $\sin 2\theta \approx 10^{-5}$ for $\delta m^2 = -1$, could be about 10^6 .

For $10^{-5} < \sin 2\theta < 10^{-3}$ a solution of eq. (51) without back-reaction results in a huge rise of the asymmetry, however, the back-reaction efficiently kills this rise and the asymmetry may increase at most by 6 orders of magnitude. It confirms the early assertion (based on oversimplified arguments) of ref. [6] that back-reaction does not permit the asymmetry to grow too much. However, the magnitude of the generated lepton asymmetry the we found is much larger than that advocated in ref. [6] but still much smaller than in refs. [8]–[11], except for a very large mass difference, $\delta m^2 \approx 10^9$ eV, where the asymmetry may be above 0.1. We have solved eq. (51) numerically in the above mentioned range of $\sin 2\theta$. For sufficiently small values of q the equation was solved directly without any simplifications, while for larger q , when the product $\zeta = 3.3 \cdot 10^{-3} Z (\cos 2\theta \delta m^2)^{-1/3}$ became close to unity, the integral was estimated in the resonance approximation. There are two resonances corresponding to the condition

$D^2 = U^2$ that give opposite sign contributions to the integral, and eq. (51) becomes:

$$\zeta' = \sum_{j=1,2} \frac{\pi B}{2} b_0 (1/t_j) \frac{q^{8/3}(t_j^2 - 1) f_{eq}(qt_j)}{\zeta \sqrt{4q^{2/3} + \zeta^2}} \left[1 - B_1 \frac{(t_j^2 - 1)^2}{\delta^2 t_j^4} \right], \quad (55)$$

where $t_j = \pm \zeta/2q^{1/3} + \sqrt{1 + (\zeta/2q^{1/3})^2}$.

For $\delta m^2 = -1$ we present the evolution of η as a function of the decreasing temperature T for 3 different mixing angles in fig. 2, where the lepton asymmetry is $\eta = |\eta_B/4 - (n_{\nu_e} - n_{\bar{\nu}_e})/n_\gamma|$. The solid line is for $\sin 2\theta = 1 \cdot 10^{-5}$, and it is clearly seen that the asymmetry is frozen at a very low value. For bigger mixing angles, $\sin 2\theta = 2 \cdot 10^{-5}$ (dashed) or $\sin 2\theta = 3 \cdot 10^{-4}$ (dotted) the increase may be much bigger. The dotted line clearly shows a power law behavior, $\eta \propto T^{-1}$, following the exponential increase. If one neglects the back-reaction, $B_1 = 0$, then the power law becomes $\eta \propto T^{-11/3}$.

In fig. 3 we plot the final value of η as a function of $\sin 2\theta$ for $\delta m^2 = -1$. One clearly sees the sharp exponential cut-off around $\sin 2\theta = 10^{-3}$. The final lepton asymmetry in the region with a large increase, $10^{-5} < \sin 2\theta < 10^{-3}$, does not depend on the initial value, η_{in} , whereas the final value of η is almost linear in η_{in} for $\sin 2\theta < 10^{-5}$.

For smaller masses the region of increase is shifted slightly to higher mixing angles as is seen in fig. 4, where we plot the final lepton asymmetry, η , as functions of $\sin 2\theta$ for 5 different masses, $-\delta m^2 = 10^{-6}, 1, 10^6, 10^9, 10^{12}$. This shift is caused by $B \sim (|\delta m^2|)^{1/6}$ (see eq. (52)). On the other hand, for bigger masses the exponential cut-off moves to smaller mixing angles. This is because Δb_0 goes like $\sqrt{|\delta m^2|}$. Clearly the effect with the biggest masses has limited applicability, since very heavy particles would become non-relativistic early, however, it is comforting to note that the maximal asymmetry generated does not continue rising for very heavy neutrinos. The region of instability in the $(\delta m^2, \sin 2\theta)$ -plane is presented in fig. 5.

Our eq. (51) is very similar to the equations describing the evolution of the asymmetry derived in ref. [19]. However, we took terms related to the variation of b_0 into account and these terms are responsible for the stabilization of the rise of the

asymmetry when the latter is still small. Our derivation of the evolution equation is somewhat different and to our mind it is more rigorous.

The result are only valid in the case of small mixing angles, $\sin 2\theta < \delta = 1/135$. In the other limiting case the evolutionary equation is quite different as well as the behavior of the asymmetry. Our preliminary results show that in the case of a large mixing angle the asymmetry does not rise. This agrees with the result found in this work that for $\sin 2\theta > 10^{-3}$ the asymmetry diminishes. Even though the asymmetry remains small, the impact of neutrino oscillations on the primordial nucleosynthesis could be non-negligible due to the effects of non-equilibrium neutrino distribution function. We will calculate the abundances of light elements in a subsequent paper.

Thus, according to our calculations, lepton asymmetry in the sector of active neutrinos may indeed be strongly enhanced. However the enhancement that we found is considerably weaker than that found in the earlier papers [8]-[10]. Even for a very large mass difference $-\delta m^2 = 10^6$ the resulting asymmetry is below 10^{-2} and only for $-\delta m^2 = 10^9$ it reaches the values that may be important for nucleosynthesis (see fig. 4). We believe that our calculations are very accurate. The only approximation is an expansion in inverse powers of Q (eq. (29)) and in powers of a small $\sin 2\theta < 0.01$. Otherwise our calculations are exact. In some cases we used analytic solutions to appropriate differential equations but in all the cases numerical solutions were quite simple and they agree very well with the analytic results. In the limit of a small δm^2 , e.g. $-\delta m^2 = 10^{-7}$ the parameter $Q \approx 18$ still remains large. Our results for such small δm^2 are in a reasonable agreement with direct numerical calculations made in refs. [16] and with our own ones (unpublished).

We see that for some values of the mixing angle the asymmetry Z first very quickly goes to zero reaching extremely small values about or below 10^{-100} and later started to rise up to 10^{-5} or even somewhat larger. If one solves eq. (51) the sign of the final asymmetry remains fixed and is completely determined by the initial conditions. So in this approximation we do not observe any chaoticity in agreement with earlier

papers [9, 14]. On the other hand if one applies a direct numerical approach, then it is practically evident that chaoticity must be observed because no direct computational procedure is able to maintain an accuracy at the level 10^{-100} . Thus one would expect that in the region when the asymmetry is quite small its numerically calculated sign is arbitrary and chaotic; it is just numerical errors. When the asymmetry starts to rise its final sign is the same as the initial one at the moment when the asymmetry becomes larger than the numerical accuracy. It could possibly explain the chaoticity observed in the papers [11, 12, 13].

However one should remember that eq. (51) is valid only if Z does not decrease too fast with increasing q . So for a small Z one cannot say on the basis of eq. (51) that the asymmetry is as small as 10^{-100} . There are some more terms in eqs. (25-28), as e.g. $F(A - S)/2$, that should not be neglected if Z vanishes. Our preliminary results show that the solution of the kinetic equations in the limit of small Z does not show any chaoticity, though the sign of Z may be different from the initial one. We do not observe the sign change but this statement demands some further checks.

There is a physically interesting possibility of chaoticity, namely if the asymmetry, as calculated through kinetic equations, is extremely small, the statistical fluctuations would be essential. The relative magnitude of a statistical fluctuation in a volume with N particles is about $1/\sqrt{N}$. So if this value is larger than the asymmetry Z the fluctuations would dominate and the sign of the asymmetry would be determined by statistical fluctuations. However, to be essential the size of the region with such a fluctuation should be larger than the neutrino diffusion length during the characteristic time of oscillations. The complexity of the calculations in such a case would increase very much because now one has to take into account the effects of the fluctuating medium on the oscillations.

A possible explanation of the difference between our results and the results of other groups (they also disagree between themselves, in particular in a possible chaotic behavior of the asymmetry) is that in most cases an assumption of kinetic equilibrium

of neutrinos was made. This assumption enormously simplifies the numerical calculations but may be strongly violated. Its violation may be crucial for the strength of generation of lepton asymmetry. We checked in a simplified example that in the opposite limit when the spectrum of neutrinos never recovered its equilibrium distribution and the resonance is complete (i.e. for relatively large $\sin 2\theta$), the asymmetry experiences only a very mild enhancement [17].

However in some papers (see e.g. [10]) it is stated that a complete set of kinetic equations was numerically solved without any approximations. It is always difficult to find the source of disagreement, especially in numerical works. As we understood from the paper [10], the system of $8N$ kinetic equations (where N is the number of points in the momentum grid) for ν and $\bar{\nu}$ (equivalent to our eqs. (14)–(17)) was solved numerically but an additional equation for the evolution of lepton asymmetry was introduced (Z in our notations and L in notations of the quoted paper). The latter was obtained from the expression for the asymmetry by differentiating the corresponding integrand containing elements of the density matrix. This equation was also solved numerically step by step and the resulting asymmetry was substituted into the equations describing the evolution of the density matrix elements. Possibly this technical trick helped to diminish the computational instability of the original equations. To calculate the integral the authors estimated it close to the resonance and integrated over the range of 3.5 resonance widths. We repeated a similar procedure for the derivative of the asymmetry Z' in the resonance approximation and found both analytically and numerically that the result strongly depends upon the integration limits. The larger are the limits the weaker is the rise of asymmetry. The integral becomes saturated when the limits are larger than 10 or even 20 widths. Such a slow rate of saturation is related to the fact that the derivative of the resonance is a function that changes sign near the resonance. Another source of disagreement may be that the integration over momentum performed in ref. [10] was symmetric around the resonance, however in reality the range of integration in negative and positive

directions are not the same (because the momentum runs from 0 to ∞). The term of the lowest order in the resonance width in the integrand is an odd function near the resonance so the contribution of asymmetry in the integration range is enhanced by $1/\delta$. This effect is not strong in the case when the lepton asymmetry remains small and its back-reaction is not essential but when it starts to rise, the asymmetry in integration limits should be taken into account. These two effects may possibly explain the difference between our results and the calculations of ref. [10]. However in our discussion with R. Foot and R. Volkas they defended stability of their calculations with respect to the choice of the region of integration. So at the moment the question about the precise origin of our disagreement remains open.

In conclusion, we have analytically transformed the complete set of momentum dependent equations governing the evolution of the neutrino distribution functions to a form which allows a simple numerical solution. The only approximation is an expansion in the small parameter $\sin 2\theta$. These equations can even be solved analytically in the limit of large Q , allowing us to derive a simple first order differential equation for the evolution of the lepton asymmetry. This differential equation takes into account the strong back-reaction effects on the generation of the lepton asymmetry due to the presence of an extra term (proportional to B_1) which is absent in approximate equations derived in some other papers. Due to this back-reaction we find that the asymmetry rise terminates at a much smaller magnitude.

Acknowledgments

The work of AD and SH was supported in part by the Danish National Science Research Council through grant 11-9640-1 and in part by Danmarks Grundforskningsfond through its support of the Theoretical Astrophysical Center. SP was supported by SISSA and by the TMR network grant ERBFMRX-CT96-0090. DS was supported in part by the Russian Foundation for Fundamental Research through grant 98-02-17493-A. DS and SP thank TAC for hospitality when part of this work was done. We acknowledge our discussion with P. Di Bari and R.R. Volkas during COSMO-99

where this work was presented. We exchanged several e-mail letter with P. Di Bari, R. Foot, and R. Volkas and though we did not reach an agreement, the discussion is very much appreciated.

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Figure Captions:

Fig. 1 b_j as functions of q for momentum $y = 1$, $\sin 2\theta = 10^{-3}$ and $\delta m^2 = -1$. The long-dashed line is $1 + b_0$, the full line is b_1 , the dotted and the dashed curves are absolute values of b_2 and b_3 respectively.

Fig. 2 The evolution of η as a function of the decreasing temperature T in MeV. The full line is for $\sin 2\theta = 1 \cdot 10^{-5}$, the dashed line is for $\sin 2\theta = 2 \cdot 10^{-5}$, and the dotted line is for $\sin 2\theta = 3 \cdot 10^{-4}$. All with $\delta m^2 = -1$.

Fig. 3 The final value of η as a function of $\sin 2\theta$ for $\delta m^2 = -1$. For mixing angles bigger than $\approx 10^{-3}$ the final value of η is exponentially suppressed (see text). We have used $\delta m^2 = -1$, $\eta_{in} \sim 10^{-10}$ and q runs from 10^{-2} to 10^3 .

Fig. 4 The final value of η as a function of $\sin 2\theta$ for 5 different masses: $-\delta m^2 = 10^{-6}$ (solid), 1 (dashed), 10^6 (dotted), 10^9 (dash-dot), and 10^{12} (long-dashed).

Fig. 5 Stability regions in $\sin 2\theta - \delta m^2$ space. Here *instability* means that the final η is more than an order of magnitude bigger than the initial η .

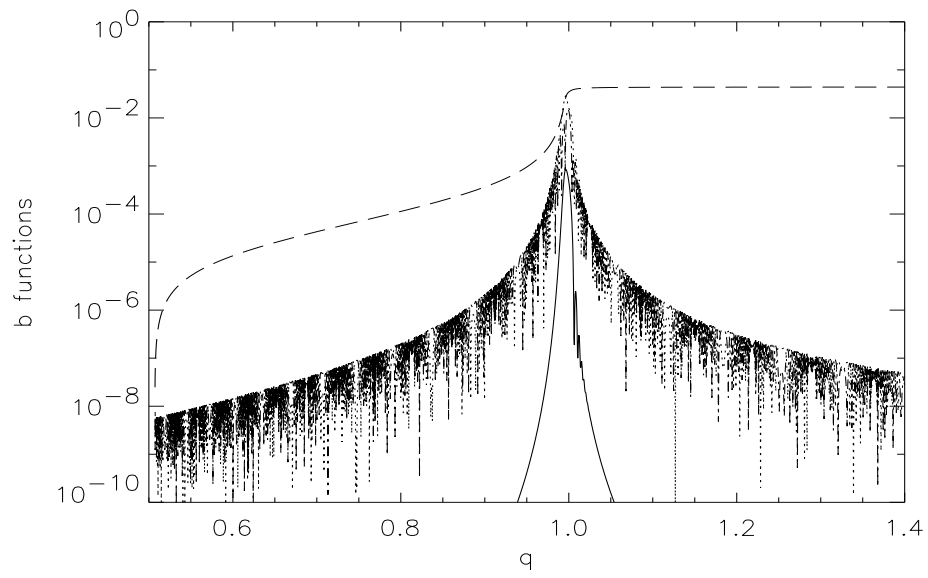


Figure 1:

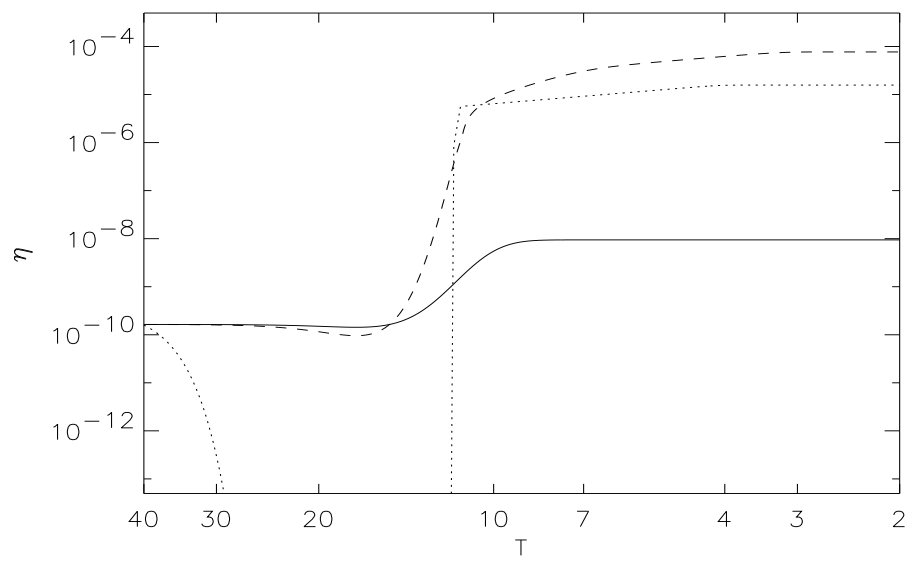


Figure 2:

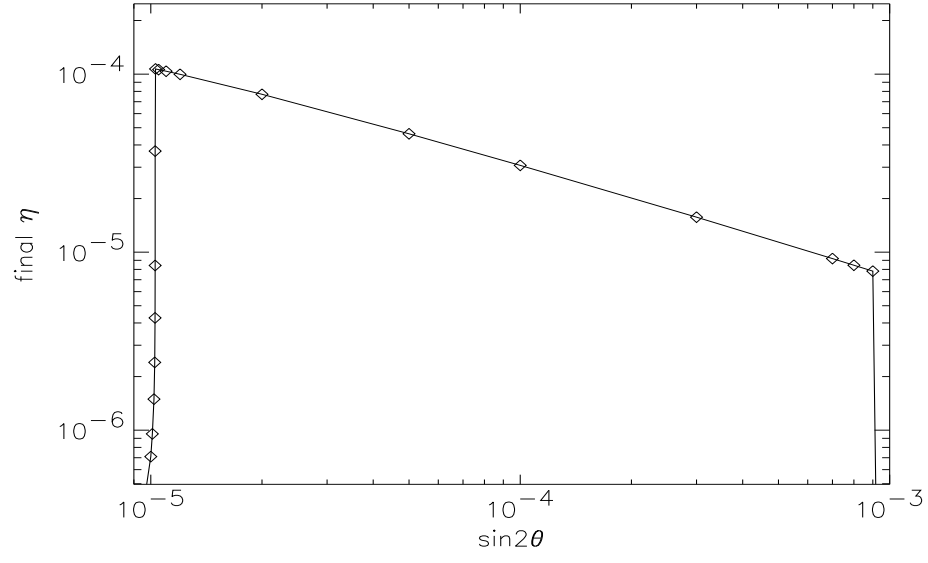


Figure 3:

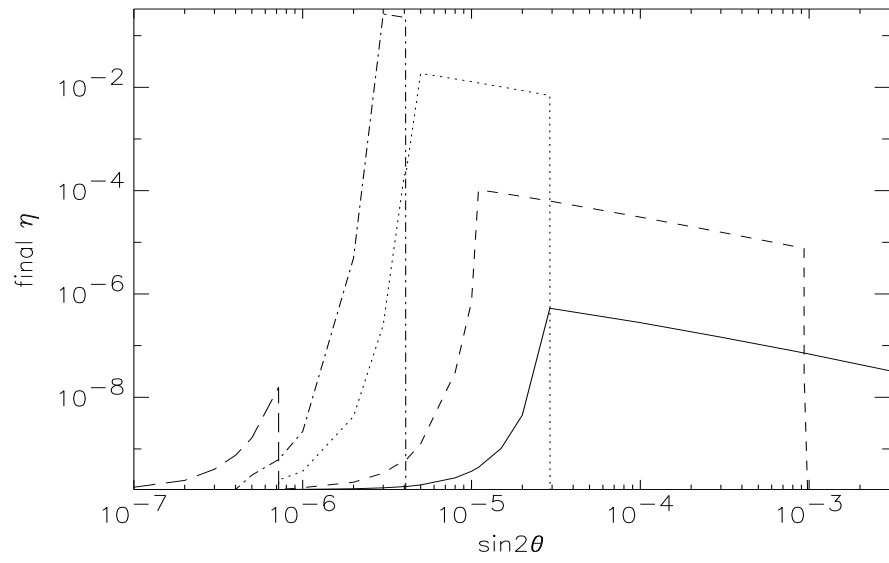


Figure 4:

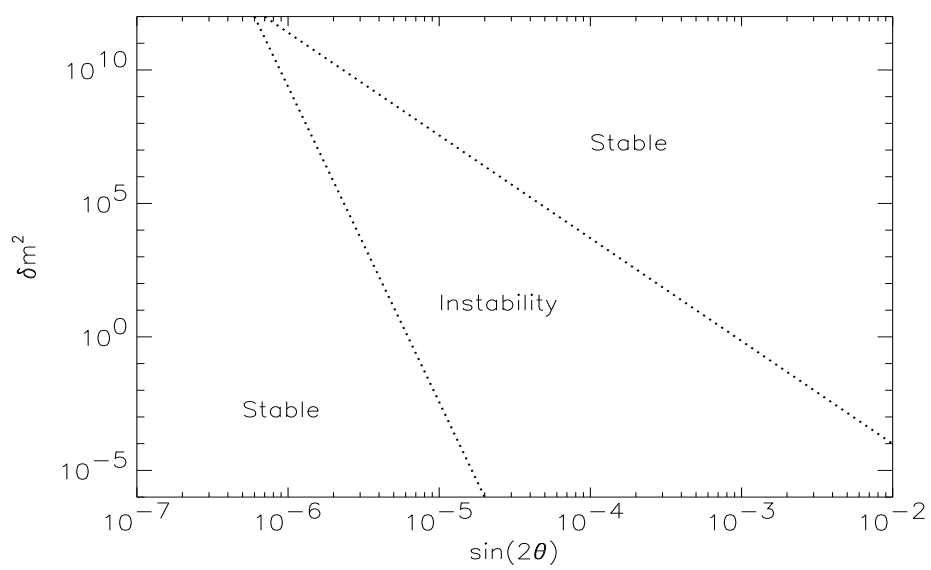


Figure 5: